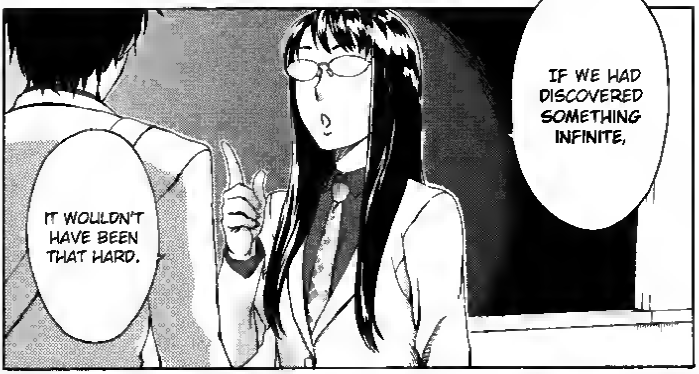
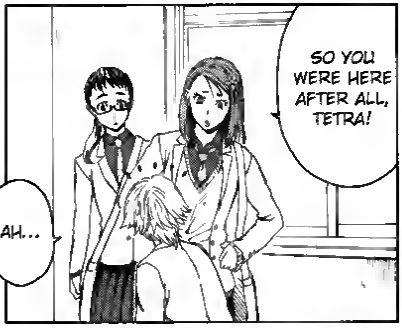
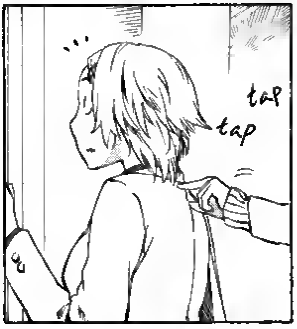
DANG
DONG
I FOUND AN INTERESTING WAY OF PROOF.
DING
DONG
FOR THE QUESTION "ON A UNIT CIRCLE WHOSE CENTER IS THE ORIGIN, DO INFINITELY MANY RATIONAL POINTS EXIST?"
POKE OUT
step
step
step
step
THE SOLUTION OF MURAKI-SENSEI'S——

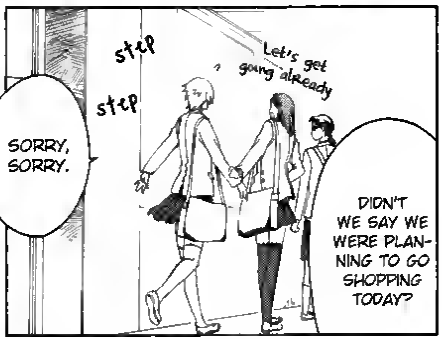
第5話


IF WE HAD DISCOVERED SOMETHING INFINITE,
IT WOULDN'T HAVE BEEN THAT HARD.


SO YOU WERE HERE AFTER ALL, TETRA!
AH...


tap
tap

NOW THEN.
SHOVE
Eh
THUD


step
step
Let's get going already
SORRY, SORRY.
DIDN'T WE SAY WE WERE PLANNING TO GO SHOPPING TODAY?

On a unit circle whose center is the origin, do infinitely many rational points exist?

FIRST WE RECONFIRM THE QUESTION

$x^2 + y^2 = 1$

WE WILL SUPPOSE (x, y) ARE THE POINTS OF THE COORDINATE PLANE.

y

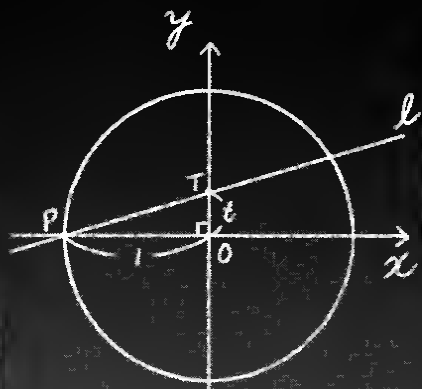
x

THE PROPOSITION THAT INFINITELY MANY RATIONAL POINTS EXIST ON THIS CIRCLE IS

THE EQUATION REPRESENTING A CIRCLE WITH CENTER AT THE ORIGIN AND RADIUS ONE WILL BE

EQUIVALENT TO THE PROPOSITION THAT THE EQUATION $x^2 + y^2 = 1$

HAS INFINITELY MANY RATIONAL SOLUTIONS.



LIKE NOW, THROUGH POINT $P(-1, 0)$ WHICH IS ON THE CIRCUMFERENCE

WE DRAW THE STRAIGHT LINE l WITH SLOPE t.

We cut the unit circle with the straight line l.

$y = tx + t$

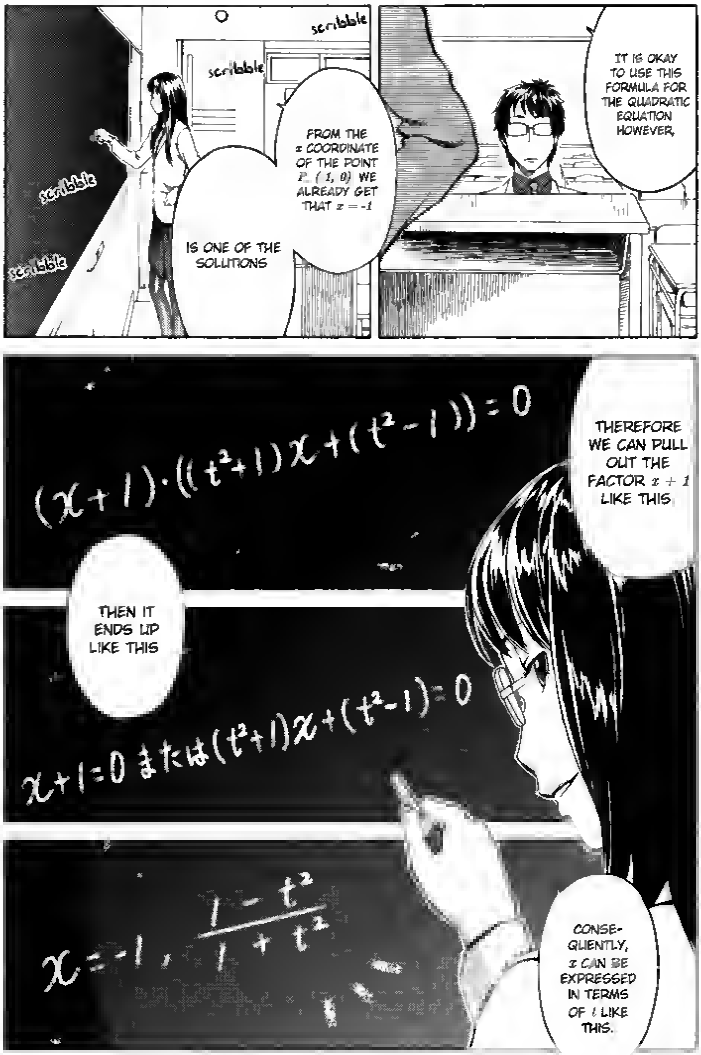
SINCE ITS SLOPE IS t, AND IT PASSES THROUGH THE POINT $T(0, 1)$, THE EQUATION OF THE STRAIGHT LINE l WILL BE LIKE THIS.

LET'S CALL THAT POINT Q.

IF WE EXCLUDE THE STRAIGHT LINE l TOUCHING POINT P,

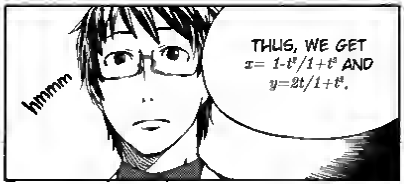
THERE MUST BE ONE OTHER POINT BESIDES P WHERE THE STRAIGHT LINE l AND THE CIRCLE INTERSECT.

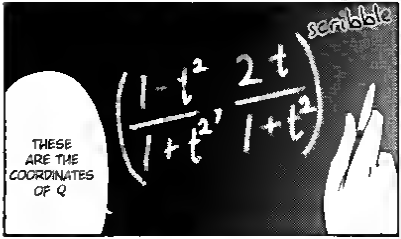
$\begin{cases} x^2+y^2=1 & \text{equation of the circle.} \\ y=tx+t & \text{equation of the straight line } l. \end{cases}$
TO EXPRESS Q IN TERMS OF t, WE CAN USE THE FOLLOWING SIMULTANEOUS EQUATIONS.
THAT'S BECAUSE SOLUTIONS OF SIMULTANEOUS EQUATIONS CORRESPOND TO POINTS OF INTERSECTION IN THE DIAGRAM REPRESENTING THE EQUATIONS
$x^2+y^2=1$ equation of the circle
$x^2+(tx+t)^2=1$ substitute in $y=tx+t$
$x^2+t^2x^2+2t^2x+t^2=1$ expand
$x^2+t^2x^2+2t^2x+t^2-1=0$ move the 1
$(t^2+1)x^2+2t^2x+t^2-1=0$ collect x^2 terms
SINCE $t^2+1\neq 0$,
THIS WILL BE A QUADRATIC EQUATION IN x.

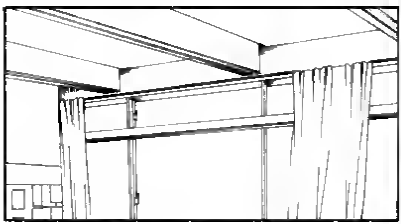

scribble
scribble
scribble
scribble
FROM THE x COORDINATE OF THE POINT P. (1, 0) WE ALREADY GET THAT x = -1
IS ONE OF THE SOLUTIONS
IT IS OKAY TO USE THIS FORMULA FOR THE QUADRATIC EQUATION HOWEVER,
$(x+1)\cdot((t^2+1)x+(t^2-1))=0$
THEREFORE WE CAN PULL OUT THE FACTOR $x+1$ LIKE THIS
THEN IT ENDS UP LIKE THIS
$x+1=0$ または $(t^2+1)x+(t^2-1)=0$
$x=-1, \frac{1-t^2}{1+t^2}$
CONSE-QUENTLY, x CAN BE EXPRESSED IN TERMS OF t LIKE THIS.

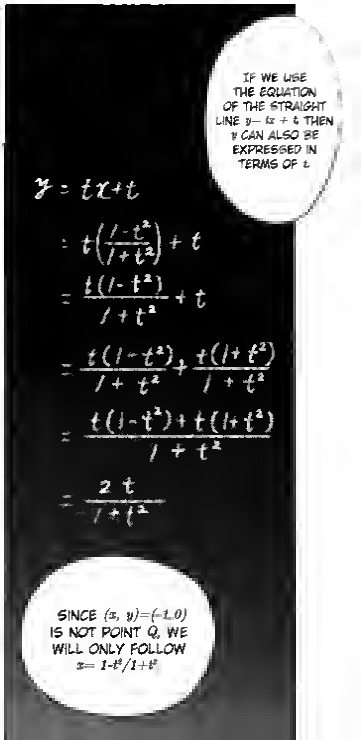
Alright
THUMBS UP

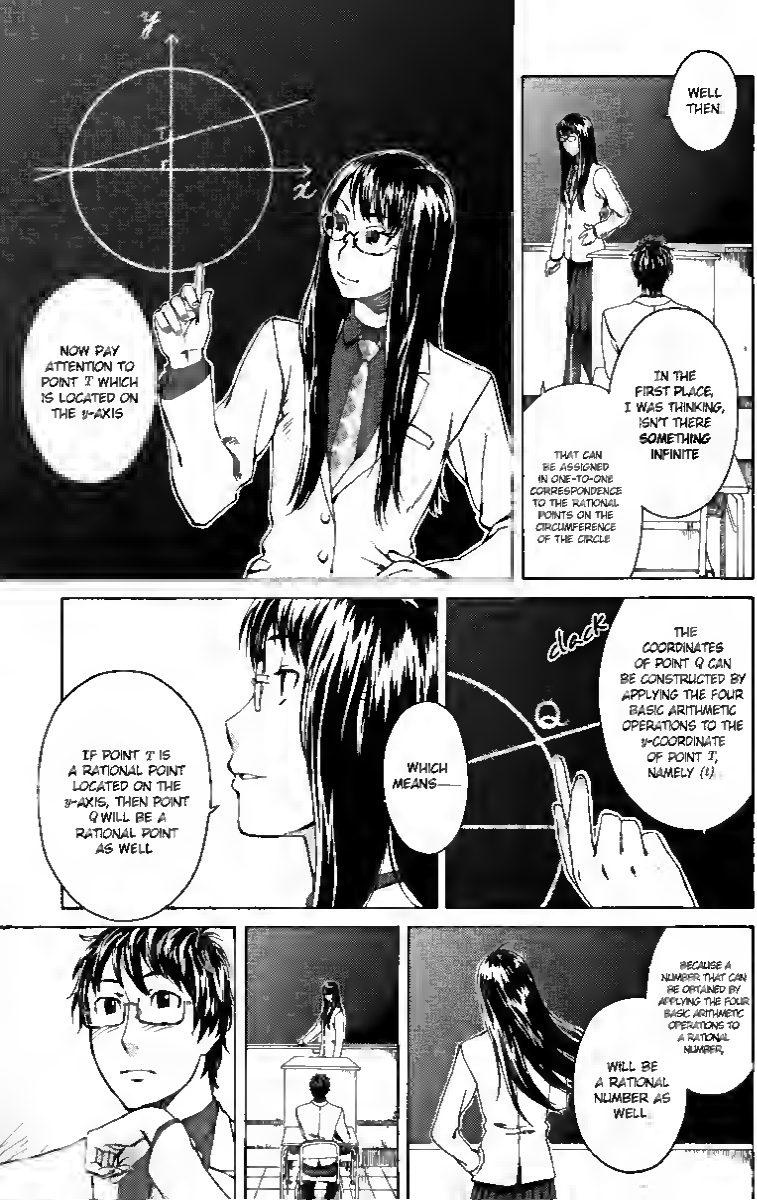
DO YOU UNDERSTAND SO FAR?

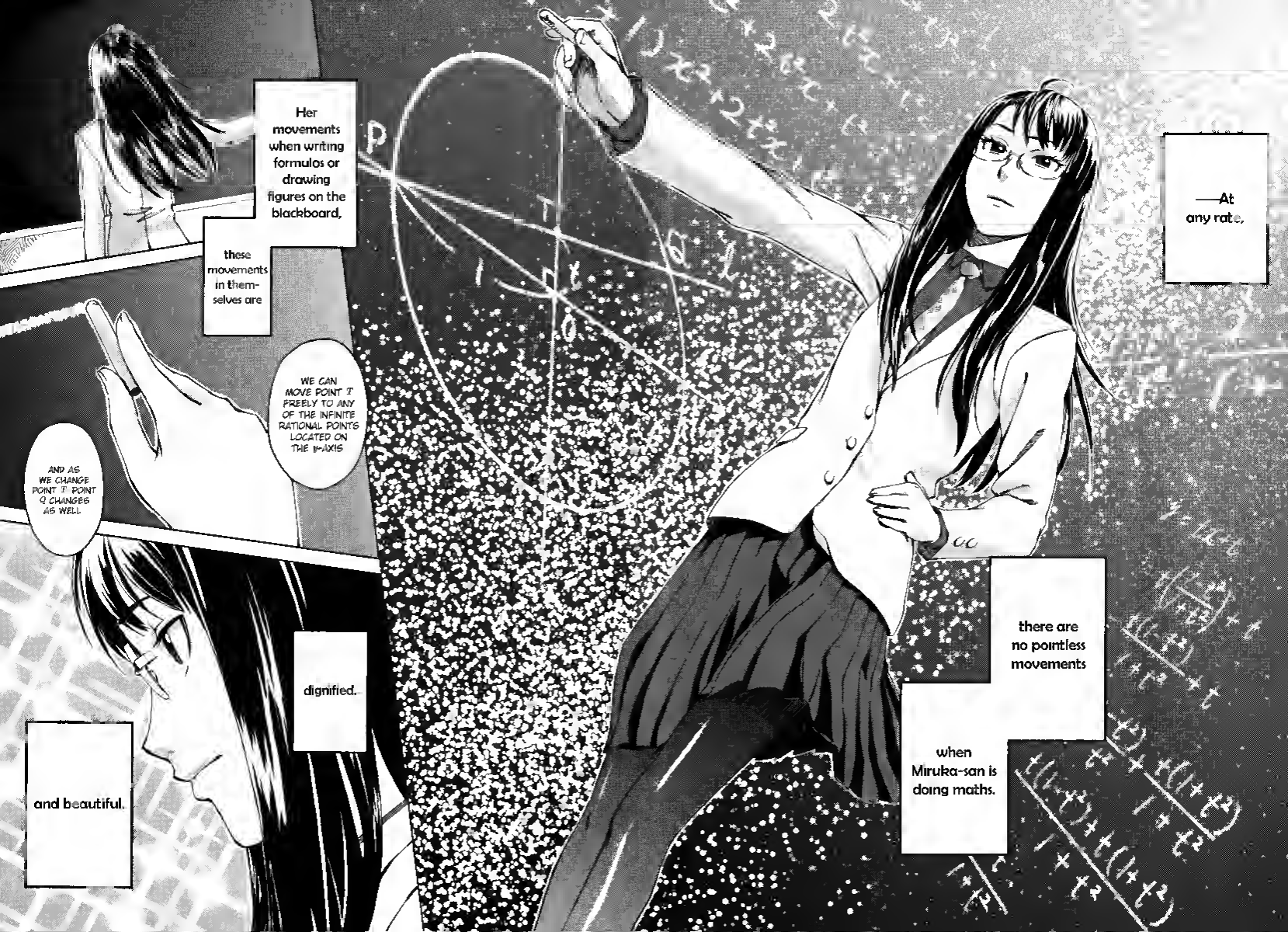

hmmm
THUS, WE GET $x = 1-t^2/1+t^2$ AND $y = 2t/1+t^2$.


scribble
$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$
THESE ARE THE COORDINATES OF Q




IF WE USE THE EQUATION OF THE STRAIGHT LINE $y = tx + t$, THEN y CAN ALSO BE EXPRESSED IN TERMS OF t
$y = tx + t$
$= t\left(\frac{1-t^2}{1+t^2}\right) + t$
$= \frac{t(1-t^2)}{1+t^2} + t$
$= \frac{t(1-t^2)}{1+t^2} + \frac{t(1+t^2)}{1+t^2}$
$= \frac{t(1-t^2)+t(1+t^2)}{1+t^2}$
$= \frac{2t}{1+t^2}$
SINCE $(x, y) = (-1, 0)$ IS NOT POINT Q, WE WILL ONLY FOLLOW $x = 1-t^2/1+t^2$


y
x
T
NOW PAY ATTENTION TO POINT T WHICH IS LOCATED ON THE y-AXIS
WELL THEN.
IN THE FIRST PLACE, I WAS THINKING, ISN'T THERE SOMETHING INFINITE
THAT CAN BE ASSIGNED IN ONE-TO-ONE CORRESPONDENCE TO THE RATIONAL POINTS ON THE CIRCUMFERENCE OF THE CIRCLE
dark
Q
THE COORDINATES OF POINT Q CAN BE CONSTRUCTED BY APPLYING THE FOUR BASIC ARITHMETIC OPERATIONS TO THE y-COORDINATE OF POINT T, NAMELY (t).
WHICH MEANS—
IF POINT T IS A RATIONAL POINT LOCATED ON THE y-AXIS, THEN POINT Q WILL BE A RATIONAL POINT AS WELL
BECAUSE A NUMBER THAT CAN BE OBTAINED BY APPLYING THE FOUR BASIC ARITHMETIC OPERATIONS TO A RATIONAL NUMBER,
WILL BE A RATIONAL NUMBER AS WELL.


——At any rate,
there are no pointless movements
when Miruka-san is doing maths.
Her movements when writing formulas or drawing figures on the blackboard,
these movements in themselves are
dignified.
and beautiful.
WE CAN MOVE POINT T FREELY TO ANY OF THE INFINITE RATIONAL POINTS LOCATED ON THE Y-AXIS
AND AS WE CHANGE POINT T POINT Q CHANGES AS WELL

OF COURSE!

OH!

HEY! ARE YOU LISTENING?

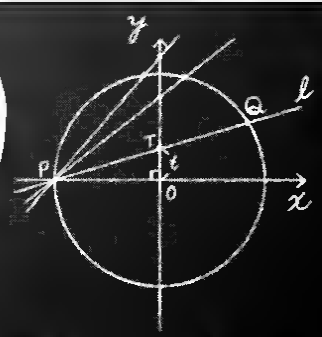
hmm

Answer

in a unit circle whose centre is the origin, infinitely many rational points do exist.

WE MANAGED TO DEMONSTRATE THAT

INFINITELY MANY RATIONAL POINTS DO EXIST ON THE CIRCUMFERENCE OF THIS UNIT CIRCLE.



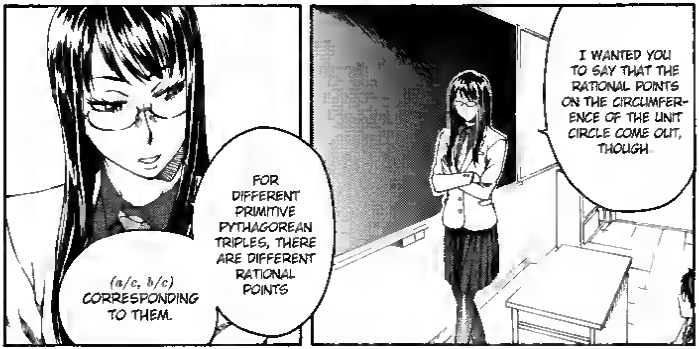
FROM THE PREVIOUS...

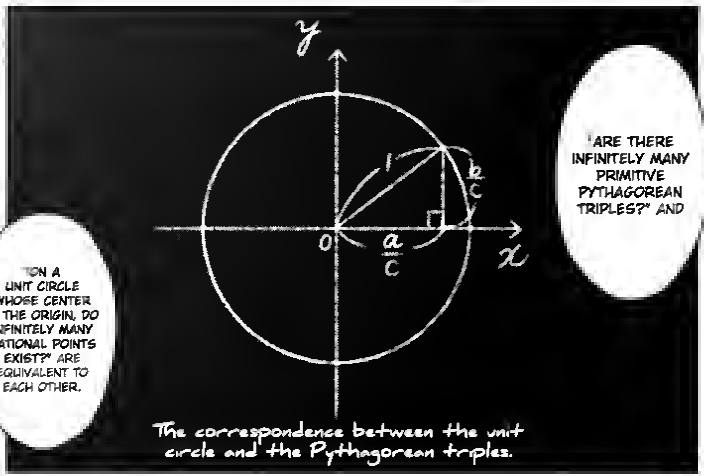
Shifting point T shifts point Q.

I SEE...
YOU HAVEN'T NOTICED YET?
WHAT?
YOU ARE REMARKABLY SLOW TODAY
I'M TALKING ABOUT TETRA
adjust
!
W, WE WEREN'T TOGETHER AT LUNCH
I DIDN'T ASK ABOUT THAT

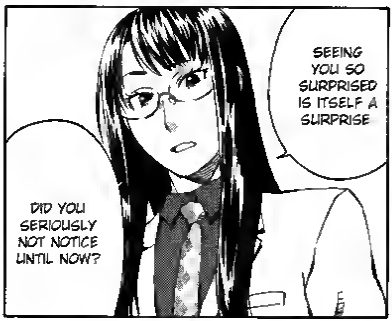
DIDN'T YOU SEE TETRA'S CARD?
EH...?
THAT WILL BE $(a/c)^2+(b/c)^2=1$...
THINK OF THE PYTHAGOREAN THEOREM $a^2+b^2=c^2$, WHERE a, b, c ARE NATURAL NUMBERS
IF WE DIVIDE BOTH SIDES BY c^2,
WHAT WILL COME OUT?

AH!
$(x,y)=(a/c,\ b/c)$ IS A SOLUTION OF $x^2+y^2=1$!
SO A UNIT CIRCLE COMES OUT OF THE PYTHAGOREAN THEOREM!


I WANTED YOU TO SAY THAT THE RATIONAL POINTS ON THE CIRCUMFERENCE OF THE UNIT CIRCLE COME OUT, THOUGH.
FOR DIFFERENT PRIMITIVE PYTHAGOREAN TRIPLES, THERE ARE DIFFERENT RATIONAL POINTS
$(a/c,\ b/c)$ CORRESPONDING TO THEM.


y
x
0
1
b/c
a/c
"ARE THERE INFINITELY MANY PRIMITIVE PYTHAGOREAN TRIPLES?" AND
"ON A UNIT CIRCLE WHOSE CENTER IS THE ORIGIN, DO INFINITELY MANY RATIONAL POINTS EXIST?" ARE EQUIVALENT TO EACH OTHER.
The correspondence between the unit circle and the Pythagorean triples.

!!
WHICH MEANS, THE TWO CARDS WERE BASICALLY THE SAME PROBLEM.


SEEING YOU SO SURPRISED IS ITSELF A SURPRISE
DID YOU SERIOUSLY NOT NOTICE UNTIL NOW?

I DIDN'T NOTICE...
On a unit circle whose center is the origin, do infinitely many rational points exist?
Are there infinitely many primitive Pythagorean triples?
IN MIRUKA-SAN'S CARD, A RELATION OF RATIONAL NUMBERS WAS WRITTEN
ON TETRA-CHAN'S CARD, A RELATION OF INTEGERS WAS WRITTEN
EVEN THOUGH I SAW THEM...!
EVEN THOUGH I SAW BOTH CARDS...

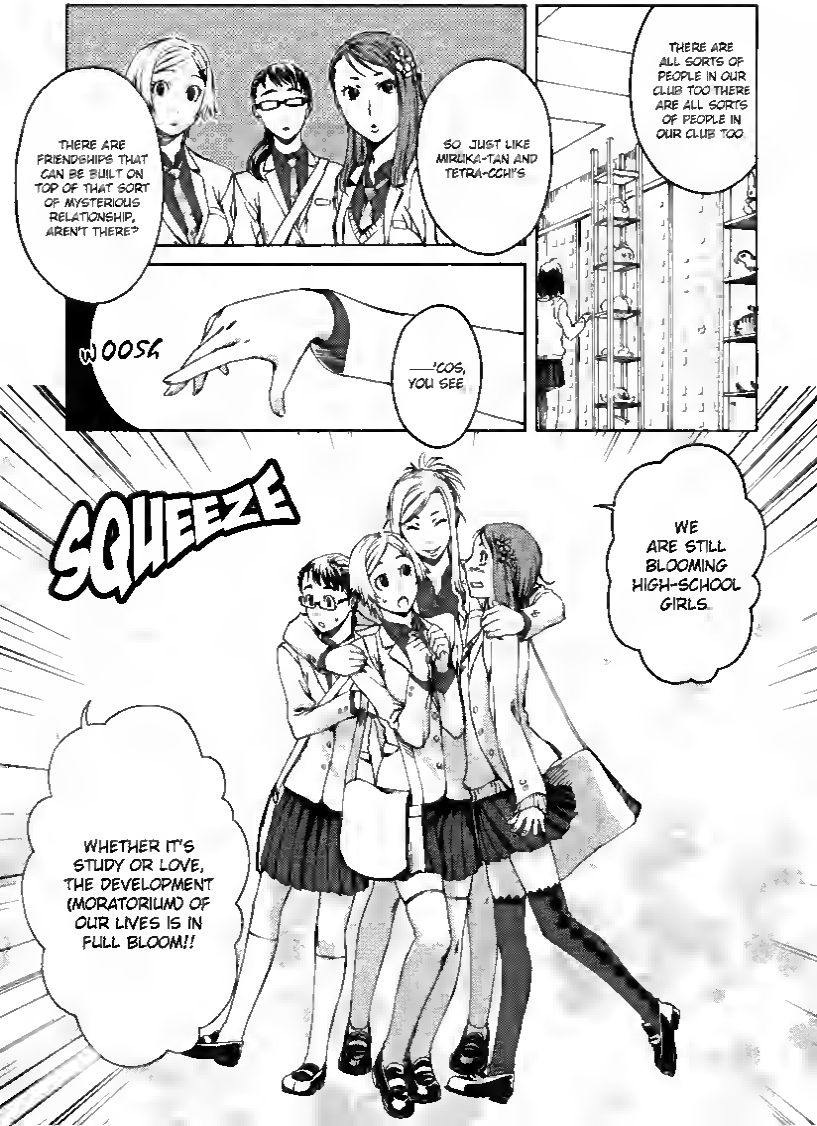
BA-DUMP
IT WILL BE TROUBLESOME IF YOU GET DEPRESSED OVER SOMETHING LIKE THAT.
USING COMBINED CARDS LIKE THIS IS MURAKI-SENSEI'S USUAL TRICK, YOU KNOW?
FUWO
I can sense Muruka-san's smell
WITH THOSE TWO CARDS, SENSEI HINTED AT A MYSTERY.
"PERCEIVING THINGS AS DIAGRAMS" IS THE THEME OF GEOMETRY
It's the smell of citrus.
"LOOKING FOR SOLUTIONS OF EQUATIONS" IS THE THEME OF ALGEBRA.

ALGEBRA AND GEOMETRY——
SURROUNDED BY THAT BITTERSWEET, REFRESHING SMELL...
MY HEART MYSTERIOUSLY CALMED DOWN.
PERHAPS MURAKI-SENSEI WANTED TO SHOW US THOSE TWO WORLDS.

TWO WORLDS...
THOSE WHO COUNT THE STARS ... THOSE WHO DRAW THE CONSTELLATIONS,
WHICH ONE IS ONII-CHAN?

Cuty Rabbit.
SAY TETRA, WHY ARE YOU OKAY WITH THOSE GUYS?
WELL, AS FOR THAT SENPAI YOU ALMOST CHASED TO HIGH SCHOOL, I KNOW THAT YOU LIKE HIM, BUT STILL,
serious
WHY ARE YOU
ACTING FRIENDLY EVEN WITH A RIVAL WHO'S THAAAT MUCH SCARY!?

WEDDING
POKES OUT
HOW MUCH IS THAAAT MUCH?
SMILE
'ELLO, I'M MIRUKA-TAN'S BUDDY (LOVER), AY-AY.
CRAAAAP —!!
Hm!
Hm, Tetsuka-chi
BACKS
AWAY
WELL.
THAT'S FINE
IT'S NATURAL TO THINK THAT WAY TOO.


THERE ARE ALL SORTS OF PEOPLE IN OUR CLUB TOO THERE ARE ALL SORTS OF PEOPLE IN OUR CLUB TOO.
SO JUST LIKE MIRUKA-TAN AND TETRA-CCHI'S
THERE ARE FRIENDSHIPS THAT CAN BE BUILT ON TOP OF THAT SORT OF MYSTERIOUS RELATIONSHIP, AREN'T THERE?
WOOSH
—'COS, YOU SEE
SQUEEZE
WE ARE STILL BLOOMING HIGH-SCHOOL GIRLS.
WHETHER IT'S STUDY OR LOVE, THE DEVELOPMENT (MORATORIUM) OF OUR LIVES IS IN FULL BLOOM!!

swish
WELL,
I DUN GET IT VERY WELL THOUGH
TETRA'S SENPAI
later!
.....
PFFT
ha ha ha ha
THEY ARE ALL WEIRD PEOPLE
THAT MIGHT BE RIGHT.
giggle

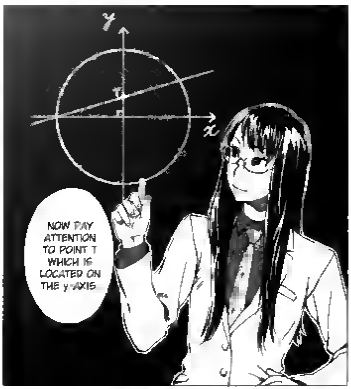
Diagrams and equations.

In mathematics there are times when we use equations or deal with diagrams.

For example, as Miruka san wrote on the blackboard, "A circle whose centre is the origin and whose radius equals 1" can be expressed by the equation "$x^2+y^2=1$". That's called the equation of the circle. If we assume the blackboard is a coordinate plane, a pair of numbers x and y, namely (x, y), will express one point on the blackboard. Let's suppose we gather all the points (x, y) that fulfill the equation of the circle $x^2+y^2=1$ and then mark them on the blackboard. If we do that, these points will collectively make a circle.

Miruka san also used an equation for a straight line, namely "y=tx+t".

The ordered pairs (x, y) that fulfill both the straight line equation and the circle equation are the points where the circle and the straight line cross each other. In other words, points of intersection. That's to say, by solving the simultaneous equations, we can obtain the points of intersection.



数学ガール

Mathematical Girls Fermat's Last Theorem

フェルマーの最終定理